

A NEW CRITERION FOR COMPLETELY MONOTONIC FUNCTIONS

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A function $f(x)$ is *completely monotonic* (c.m.) in $0 < x < \infty$ if it belongs to C^∞ and

$$(1) \quad (-1)^k f^{(k)}(x) \geq 0 \quad (k \geq 0, x > 0).$$

If $f(x)$ can be extended to be continuous at $x=0$ it is said to be c.m. in $0 \leq x < \infty$.

Various conditions are known under which a function is c.m. [4]⁽¹⁾. Bernstein proved that if

$$(2) \quad (-1)^k \Delta_h^k f(x) = \sum_0^k C_{k,n} (-1)^n f(x + nh) \geq 0 \quad (k \geq 0, x > 0, h > 0)$$

then $f(x)$ is c.m. in $0 < x < \infty$. It is known, though apparently not stated explicitly in the literature, that if we assume the continuity of $f(x)$, then we need require (2) only for some infinite sequence of integers k . (This may be obtained, for example, by use of the results of [1].) A fundamental theorem of Bernstein and Widder states that a function is c.m. in $0 < x < \infty$ if and only if it admits the representation

$$(3) \quad f(x) = \int_0^\infty e^{-xt} dF(t), \quad x > 0, F(t) \text{ increasing.}$$

A new difference criterion which includes the above is suggested by the following considerations. If $h_k = o(1/k^2)$ then (3) is inverted by [3, Theorem 4.2],

$$F(t) - F(0) = f(\infty) + \lim_{k \rightarrow \infty} d_k \int_{k/t}^\infty x^{k-1} \Delta_{h_k}^k f(x) dx,$$

where

$$(4) \quad d_k = (-h_k)^{-k} / (k-1)!.$$

This suggests the following theorem, which is the principal result of this paper.

THEOREM. *Let $f(x)$ be continuous for $x \geq 0$ and have a limit at infinity. Suppose it satisfies the inequalities*

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⁽¹⁾ Numbers in brackets refer to the references listed at the end of the paper.

$$(5) \quad (-1)^k \Delta_{hk}^k f(x) \geq 0 \quad (x > 0)$$

for an infinite sequence of integers k , where

$$h_k > 0, \quad h_k = o(1/k^2) \quad (k \rightarrow \infty).$$

Then $f(x)$ is c.m. in $0 \leq x < \infty$.

This says essentially that in the difference criteria it is sufficient that the inequalities hold for *one* (suitable) value of h for each k , rather than *all* $h > 0$.

Before proceeding to the proof we make some observations about the theorem.

(i) The conditions are trivially necessary.

(ii) The continuity of $f(x)$ is not a redundant condition. For let $\phi(x)$ be a discontinuous solution of the functional equation $\phi(x+y) = \phi(x) + \phi(y)$ [2, p. 96]. Both $\pm\phi(x)$ are convex and hence unbounded in every interval [2, pp. 91–92]. Then (5) is satisfied for arbitrarily small $\{h_k\}$ by the function $f(x) = e^{\phi(x)}$.

1. **Lemmas.** The following identity is known [4, p. 303].

LEMMA 1. For $k > 0$, $x \geq 0$, $u > 0$,

$$\frac{\partial^k}{\partial u^k} [e^{-kx/u} u^{k-1}] = \frac{1}{u} \left(\frac{kx}{u} \right)^k e^{-kx/u}.$$

For fixed x , these functions are increasing in $0 \leq u \leq x/2$.

LEMMA 2. If $g(x)$ and $r(x)$ are any functions of x then

$$\Delta_h^k [g(x)r(x)] = \sum_0^k C_{k,n} \Delta_h^n g(x + \overline{k-n}h) \Delta_h^{k-n} r(x).$$

This is the analogue of Leibniz' rule for the differentiation of a product and can be established by induction.

LEMMA 3 (GENERALIZED ROLLE'S THEOREM). If $f(x) \in C^k$ then

$$\Delta_h^k f(x) = h^k f^{(k)}(X)$$

where X lies between x and $x+kh$.

LEMMA 4. Suppose $k \geq 1$, $x > 0$, $h > 0$ are fixed, and $f(u)$ is continuous for $u \geq kh$, $f(\infty) = 0$. Then

$$\lim_{h \rightarrow 0} \int_{kh}^{\infty} \Delta_{-h}^k [e^{-\epsilon u} e^{-kx/u} u^{k-1}] f(u) du = \int_{kh}^{\infty} \Delta_{-h}^k [e^{-kx/u} u^{k-1}] f(u) du,$$

where we difference with respect to u .

Proof. The existence of the integral on the left is guaranteed by the presence of the factor $e^{-\epsilon u}$. The integral on the right converges since, by virtue of Lemmas 3 and 1, its integrand is dominated for large u by $(u - kh)^{-k-1}|f(u)|$.

If we subtract the right-hand side the problem is then to show that

$$(6) \quad H(\epsilon) = \int_{kh}^{\infty} \Delta_{-h}^k [g(u)r(u)] f(u) du = o(1) \quad (\epsilon \rightarrow 0+),$$

where

$$g(u) = e^{-kx/u} u^{k-1}, \quad r(u) = 1 - e^{-\epsilon u}.$$

Using Lemma 2 with h replaced by $-h$ and separating out the term for which $n=k$ we obtain

$$(7) \quad \begin{aligned} H(\epsilon) &= \sum_{n=0}^{k-1} C_{k,n} \int_{kh}^{\infty} \Delta_{-h}^n g(u - \overline{k-n}h) \cdot \Delta_{-h}^{k-n} r(u) \cdot f(u) du \\ &\quad + \int_{kh}^{\infty} \Delta_{-h}^k g(u) \cdot r(u) \cdot f(u) du \\ &= K(\epsilon) + L(\epsilon). \end{aligned}$$

By Lemmas 3 and 1 the integral $\int_{kh}^{\infty} \Delta_{-h}^k g(u) \cdot f(u) du$ exists. Hence

$$L(\epsilon) = \int_{kh}^{\infty} (1 - e^{-\epsilon u}) \Delta_{-h}^k g(u) f(u) du = o(1) \quad (\epsilon \rightarrow 0+).$$

We turn now to the expression $K(\epsilon)$. First by Lemma 3

$$\Delta_{-h}^{k-n} r(u) = (-h)^{k-n} r^{(k-n)}(U) = (h\epsilon)^{k-n} e^{-\epsilon U}, \quad u - (k-n)h \leq U \leq u,$$

so that for $0 < \epsilon < 1$

$$(8) \quad |\Delta_{-h}^{k-n} r(u)| \leq K_1 \epsilon^{k-n} e^{-\epsilon u},$$

where K_1 does not depend on u or ϵ . Also

$$(9) \quad \Delta_{-h}^n g(u - \overline{k-n}h) = (-h)^n g^{(n)}(U), \quad u - kh \leq U \leq u - (k-n)h.$$

But

$$g^{(n)}(u) = \sum_{j=0}^n C_{n,j} \frac{\partial^j}{\partial u^j} [e^{-kx/u}] [u^{k-1}]^{(n-j)}.$$

If $x > 0$

$$\frac{\partial^j}{\partial u^j} [e^{-kx/u}] = O(u^{-j}) \quad (u \rightarrow \infty),$$

so that

$$g^{(n)}(u) = \sum O(u^{-j} u^{k-1-n+j}) = O(u^{k-n-1}), \quad 0 \leq n \leq k-1.$$

If $x=0$ this result is obvious. Then by (9)

$$(10) \quad |\Delta_{-h}^n g(u - \overline{k-n}h)| \leq K_2 u^{k-n-1} \quad (0 \leq n \leq k-1),$$

where K_2 does not depend on u . Then by (7), (8), (10)

$$|K(\epsilon)| \leq K_1 K_2 \sum_{n=0}^{k-1} C_{k,n} \int_{kh}^{\infty} \epsilon^{k-n} e^{-\epsilon u} u^{k-n-1} |f(u)| du \quad (0 < \epsilon < 1).$$

Since $f(\infty)=0$ a simple Abelian argument proves that each term of the sum approaches zero with ϵ . Hence $K(0+)=0$, so $H(0+)=0$; this establishes (6).

LEMMA 5. Let $k \geq 1$, $h > 0$, $x \geq 0$ be fixed, $f(u)$ continuous, $0 \leq u < \infty$, $f(\infty)=0$. If $\Delta_h^k f(u)$ does not change sign in $0 \leq u < \infty$, then

$$(11) \quad \begin{aligned} & \int_0^{\infty} e^{-kx/u} u^{k-1} \Delta_h^k f(u) du \\ &= \sum_{n=0}^k C_{k,n} (-1)^{k-n} \int_{nh}^{kh} e^{-kx/(u-nh)} (u-nh)^{k-1} f(u) du \\ &+ \int_{kh}^{\infty} \Delta_{-h}^k [e^{-kx/u} u^{k-1}] \cdot f(u) du, \end{aligned}$$

where we difference with respect to u .

Clearly the integral

$$I(\epsilon) = \int_0^{\infty} e^{-\epsilon u} e^{-kx/u} u^{k-1} \Delta_h^k f(u) du$$

exists for all $\epsilon > 0$. We have

$$\begin{aligned} I(\epsilon) &= \sum_0^k C_{k,n} (-1)^{k-n} \int_0^{\infty} e^{-\epsilon u} e^{-kx/u} u^{k-1} f(u+nh) du \\ &= \sum_0^k C_{k,n} (-1)^{k-n} \left(\int_{nh}^{kh} + \int_{kh}^{\infty} \right) \exp[-\epsilon(u-nh) - kx/(u-nh)] \\ &\quad \cdot (u-nh)^{k-1} f(u) du, \end{aligned}$$

obtained by a change of variable. Then

$$(12) \quad I(\epsilon) = A(\epsilon) + B(\epsilon),$$

where

$$A(\epsilon) = \sum_0^k C_{k,n} (-1)^{k-n} \int_{nh}^{kh} \exp[-\epsilon(u-nh) - kx/(u-nh)] \cdot (u-nh)^{k-1} f(u) du,$$

$$B(\epsilon) = \int_{kh}^{\infty} \Delta_{-h}^k [e^{-\epsilon u} e^{-kx/u} u^{k-1}] f(u) du.$$

By dominated convergence

$$(13) \quad A(0+) = \sum_0^k C_{k,n} (-1)^{k-n} \int_{nh}^{kh} e^{-kx/(u-nh)} (u-nh)^{k-1} f(u) du,$$

and by Lemma 4

$$(14) \quad B(0+) = \int_{kh}^{\infty} \Delta_{-h}^k [e^{-kx/u} u^{k-1}] \cdot f(u) du.$$

From (12), (13), (14) it follows that $I(0+)$ exists and equals the expression on the right-hand side of (11). Since $\Delta_h^k f(u)$ does not change sign in $0 \leq u < \infty$, a Tauberian theorem enables us to conclude that $I(0+)$ is also equal to the left-hand side of (11) [4, p. 192].

For the remainder of the paper it is assumed that k belongs to some sequence S of non-negative integers; $k \rightarrow \infty$ means that k becomes infinite through the elements of S .

LEMMA 6. *Let $f(x)$ satisfy all the hypotheses of our theorem and suppose also that $f(\infty) = 0$. Then for any fixed $x > 0$ the quantities*

$$I_k = d_k \int_0^{\infty} e^{-kx/u} u^{k-1} \Delta_{hk}^k f(u) du$$

approach $f(x)$ as $k \rightarrow \infty$. The d_k are defined as in (4).

Proof. By Lemma 5 we have $I_k = A_k + B_k$ where

$$A_k = d_k \sum_0^k C_{k,n} (-1)^{k-n} \int_{nh_k}^{kh_k} e^{-kx/(u-nh_k)} (u-nh_k)^{k-1} f(u) du,$$

$$B_k = d_k \int_{kh_k}^{\infty} \Delta_{-h_k}^k [e^{-kx/u} u^{k-1}] f(u) du.$$

We show first that A_k vanishes with $1/k$. Let M be the maximum of $|f(x)|$ in $(0, \infty)$. Then

$$\begin{aligned} |A_k| &\leq M |d_k| \sum_0^k C_{k,n} \int_{nh_k}^{kh_k} e^{-kx/(u-nh_k)} (u-nh_k)^{k-1} du \\ &\leq M |d_k| \sum C_{k,n} \int_0^{(k-n)h_k} e^{-kx/u} u^{k-1} du \\ &\leq M |d_k| \sum C_{k,n} \int_0^{kh_k} e^{-kx/u} u^{k-1} du \\ &\leq M |d_k| e^{-x/h_k} \sum_0^k C_{k,n} \int_0^{kh_k} u^{k-1} du \\ &= M e^{-x/h_k} 2^k k^k / k!. \end{aligned}$$

Choose k_0 so that $kh_k < x/2$ for $k > k_0$. Then for $k > k_0$

$$|A_k| \leq M e^{-2k} 2^k k^k / k!.$$

By the test-ratio test this last is the general term of a convergent series, so that $A_k = o(1)$, $k \rightarrow \infty$.

We must prove then that

$$\lim_{k \rightarrow \infty} B_k = f(x).$$

By Lemmas 3 and 1, and equation (4),

$$\begin{aligned} B_k &= \frac{1}{(k-1)!} \left(\int_{x/2}^{\infty} + \int_{kh_k}^{x/2} \right) \left[\frac{1}{u} \left(\frac{kx}{u} \right)^k e^{-kx/u} \right]_{u=v-\phi_k} f(v) dv \\ &= C_k + D_k, \end{aligned}$$

where

$$(15) \quad 0 \leq \phi_k \leq kh_k.$$

By the second part of Lemma 1

$$\begin{aligned} |D_k| &\leq \frac{1}{(k-1)!} \left[\frac{1}{u} \left(\frac{kx}{u} \right)^k e^{-kx/u} \right]_{u=x/2} \int_0^{x/2} |f(v)| dv \\ &= o(1) \end{aligned} \quad (k \rightarrow \infty).$$

Our problem then is to show that $\lim_{k \rightarrow \infty} C_k = f(x)$. But it is known that [4, p. 283]

$$J_k = \frac{1}{(k-1)!} \int_{x/2}^{\infty} \frac{1}{u} \left(\frac{kx}{u} \right)^k e^{-kx/u} f(u) du \rightarrow f(x), \quad k \rightarrow \infty.$$

It therefore remains only to show that $\lim_{k \rightarrow \infty} (J_k - C_k) = 0$ and the proof of the lemma will be complete.

Now

$$(16) \quad J_k - C_k = \frac{1}{(k-1)!} \int_{x/2}^{\infty} \frac{1}{u} \left(\frac{kx}{u} \right)^k e^{-kx/u} f(u) P_k(u) du,$$

where

$$P_k(u) = 1 - \left(\frac{u}{u - \phi_k} \right)^{k+1} \exp \left(- \frac{kx\phi_k}{u(u - \phi_k)} \right) \quad (u \geq x/2).$$

We have

$$\begin{aligned} \log(1 - P_k(u)) &= (k+1) \int_{u-\phi_k}^u \frac{dv}{v} - \frac{kx\phi_k}{u(u - \phi_k)}, \\ |\log(1 - P_k(u))| &\leq \frac{(k+1)\phi_k}{u - \phi_k} + \frac{kx\phi_k}{u(u - \phi_k)}. \end{aligned}$$

By (15) we have $0 \leq \phi_k \leq kh_k$. Since $h_k = o(1/k^2)$, $\phi_k < x/4$ for $k > k_0$. It follows that for $u \geq x/2$, $k > k_0$,

$$\begin{aligned} |\log(1 - P_k(u))| &\leq \frac{(k+1)kh_k}{x/2 - x/4} + \frac{kx \cdot kh_k}{(x/2)(x/2 - x/4)} \\ &= o(1) \end{aligned} \quad (k \rightarrow \infty)$$

uniformly for $u \geq x/2$.

Let $\epsilon > 0$ be arbitrary. Then for $k > k_1$,

$$|P_k(u)| < \epsilon \quad (u \geq x/2).$$

By (16)

$$|J_k - C_k| < \frac{\epsilon}{(k-1)!} \int_{x/2}^{\infty} \frac{1}{u} \left(\frac{kx}{u}\right)^k e^{-kx/u} |f(u)| du.$$

As $k \rightarrow \infty$ the right-hand side approaches $\epsilon |f(x)|$ [4, p. 283]. Hence

$$\limsup_{k \rightarrow \infty} |J_k - C_k| \leq \epsilon |f(x)|,$$

and this completes the proof.

2. Proof of the theorem. We may assume $f(\infty) = 0$ (otherwise consider $f(x) - f(\infty)$). By hypothesis

$$L_{k,t}[f] = ((-h_k)^{-k}/k!) [x^{k+1} \Delta_{h_k}^k f(x)]_{x=k/t} \geq 0$$

for an infinite sequence of integers k . By Lemma 6 with a change of variable we have

$$f(x) = \lim_{k \rightarrow \infty} \int_0^{\infty} e^{-xt} L_{k,t}[f] dt \quad (x > 0).$$

It remains only to show that the integrals

$$(17) \quad L_k = \int_0^{\infty} L_{k,t}[f] dt$$

exist and are bounded. For then it will follow by a familiar argument that $f(x)$ has the representation (3) [4, p. 307]. From Lemma 5 with $x=0$ it follows that

$$\begin{aligned} L_k &= d_k \sum_0^k C_{k,n} (-1)^{k-n} \int_{nh_k}^{kh_k} (u - nh_k)^{k-1} f(u) du \\ &= (-h_k)^{-k} \Delta_{h_k}^k F_k(0) \end{aligned}$$

where

$$F_k(x) = \frac{1}{(k-1)!} \int_x^{kh_k} (u-x)^{k-1} f(u) du.$$

By Lemma 3

$$L_k = (-1)^k F_k^{(k)}(X_k), \quad 0 \leq X_k \leq kh_k.$$

But $F_k^{(k)}(x) = (-1)^k f(x)$, so that $L_k = f(X_k)$ and $\lim_{k \rightarrow \infty} L_k = f(0)$. Then L_k is bounded.

REFERENCES

1. R. P. Boas and D. V. Widder, *Functions with positive differences*, Duke Math. J. vol.7 (1940) pp. 496-503.
2. G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities*, Cambridge, 1934.
3. H. Pollard, *Real inversion formulas for the Laplace integral*, Duke Math. J. vol. 7 (1940) pp. 445-451.
4. D. V. Widder, *The Laplace transform*, Princeton, 1941.

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